Journal of Applied Mechanics and Technical Physics, Vol. 46, No. 3, pp. 412–422, 2005

ASYMPTOTIC BEHAVIOR OF THE CONDUCTING PROPERTIES OF HIGH-CONTRAST MEDIA

A. G. Kolpakov

UDC 519.6:517.58

The asymptotic-shielding effect and the asymptotic behavior of the conductivity of a medium containing closely spaced, perfectly conducting inclusions. It is proved that in the presence of asymptotic shielding for pairs of adjacent particles, the original continuous problem can be approximated by a finite-dimensional problem.

Key words: *shielding effect, asymptotic behavior, conductivity, perfectly conducting inclusions, finite-dimensional approximation.*

Introduction. We consider the three-dimensional boundary-value problem for the Laplace equations in a domain with perfectly conducting closely packed inclusions. Existing approaches [1–3] are inapplicable to this to problem. The history of the problem goes back to the problem of the electric field in a system of periodic bodies [4]. Later, Keller [5] showed that the formulas from [4] are inapplicable in the case of a small distance δ between the bodies and obtained new formulas for small δ. Using a linear trial function in the channel between particles, Keller [5] obtained a divergent integral. In [5], the limit of integration was arbitrary and the question of the divergence of the integral remained open. In the two-dimensional case, the corresponding integral usually converges, which made it possible to substantiate a network model for a two-dimensional composite filled with disks [3]. Tamm [6] described the shielding effect for the approach of bodies. In the present paper, it is proved that the Maxwell–Keller problem [4, 5], Tamm shielding [6], and the possibility of finite-dimensional approximation of the continuous problem in a domain with perfectly conducting inclusions are related to one another. It is found that in the three-dimensional case, the asymptotic shielding effect does not always occur and its physical nature differs from that indicated in [6].

1. Formulation of the Problem. Let nonintersecting and nonconvex particles D_i $(i = 1, 2, \ldots, N)$ with piecewise-smooth boundaries be distributed in a domain $P = [-L, L]^3$ (Fig. 1). We denote the domain outside the particles by $Q = P \setminus \{\cup D_i\}$. Let us consider the problem

$$\Delta\varphi = 0 \qquad \text{in the domain } Q; \tag{1.1}$$

$$\varphi = t_i \qquad \text{on } D_i, \quad i = 1, 2, \ldots, N; \tag{1.2}$$

$$\int\limits_{\partial D_i} \frac{\partial\varphi}{\partial\boldsymbol{n}}\, d\boldsymbol{x} = 0, \qquad i = 1, 2, \ldots, N; \tag{1.3}$$

$$\varphi(x, y, \pm 1) = \pm 1; \tag{1.4}$$

$$\frac{\partial\varphi}{\partial\boldsymbol{n}}(\pm L, y, z) = 0, \qquad \frac{\partial\varphi}{\partial\boldsymbol{n}}(x, \pm L, z) = 0. \tag{1.5}$$

Here $\boldsymbol{x} = (x, y, z)$, Δ is a Laplace operator, and $\boldsymbol{n}$ is the normal to the boundary of the domain Q. The unknowns are the function φ in the domain Q and its values $\{t_i\}$ on the particles D_i $(i = 1, 2, \ldots, N)$.

Novosibirsk State Architecturally-Building University, Novosibirsk 630008; agk@neic.nsk.su. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 46, No. 3, pp. 128–140, May–June, 2005. Original article submitted February 2, 2004; revision submitted October 25, 2004.

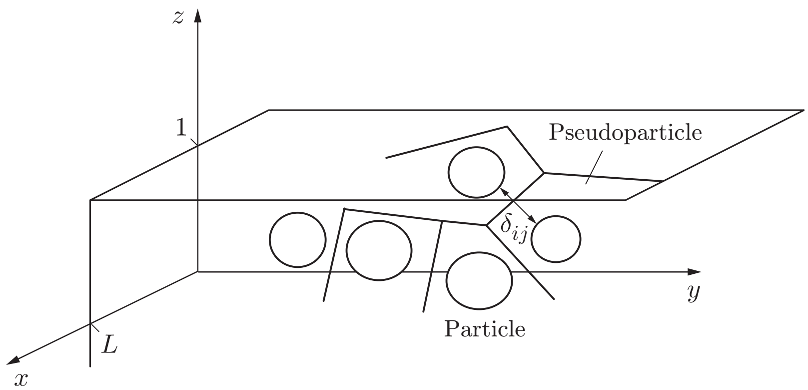

Fig. 1. Composite and Voronoi cells.

In analyzing problem (1.1)–(1.5), we shall use primarily the terminology of electrostatics. The exception is the term flux, which will be defined as $\nabla\varphi$. In electrostatics, flux is the electric-field strength with the minus sign. If (1.1)–(1.5) is treated as a problem of electrical and thermal conduction or diffusion, flux is an electric current or a heat or mass flux. In (1.1)–(1.5), the dielectric constant (resistivity, heat conductivity or diffusion) is equal to unity.

Condition (1.3) implies that the total flux in a particle is equal to zero, condition (1.4) implies that the potentials ± 1 are applied on the sides $z = \pm$, respectively (Fig. 1), and condition (1.5) implies the absence of flux through the vertical faces of the domain P.

Problem (1.1)–(1.5) is equivalent to the minimization problem

$$I(\varphi) = \frac{1}{2}\int\limits_Q |\nabla\varphi|^2\,d\boldsymbol{x} \to \min \tag{1.6}$$

on the set of functions

$$V_p = \{\varphi(\boldsymbol{x}) \in H^1(Q)\colon \varphi(\boldsymbol{x}) = t_i \text{ on } D_i,\ \varphi(x, y, \pm 1) = \pm 1\}. \tag{1.7}$$

Definition 1. The effective conductivity of an inhomogeneous medium is the quantity

$$a = \frac{1}{4L^2}\int\limits_{z=1} \frac{\partial\varphi}{\partial\boldsymbol{n}}\,d\boldsymbol{x}$$

(normal flux through the boundary $z = 1$ referred to the surface area of the boundary).

Let us express the effective conductivity in terms of the functional $I(\varphi)$. The following equality [3] holds:

$$4L^2 = \int\limits_{z=1} \frac{\partial\varphi}{\partial\boldsymbol{n}}\,d\boldsymbol{x} = \frac{1}{2}\int\limits_Q |\nabla\varphi|^2\,d\boldsymbol{x}. \tag{1.8}$$

The quantity $A = 4L^2 a$ is called the effective conductivity of the sample. By virtue of (1.8), we have

$$A = \frac{1}{2}\int\limits_Q |\nabla\varphi|^2\,d\boldsymbol{x}. \tag{1.9}$$

We investigate the effective conductivity of the sample A (and, hence, the effective conductivity a) for small distances between the particles.

2. General Upper- and Lower-Bound Estimates. *Upper-Bound Estimate.* Relations (1.6) and (1.7) imply the estimate

$$A \leqslant \frac{1}{2}\int\limits_Q |\nabla\varphi|^2\,d\boldsymbol{x} \qquad \forall\varphi \in V_p. \tag{2.1}$$

Lower-Bound Estimate. We introduce the space of functions

$$W_p = \Big\{ \boldsymbol{v} = (v_1(\boldsymbol{x}), v_2(\boldsymbol{x}), v_3(\boldsymbol{x})) \in L_2(Q): \quad \boldsymbol{v}(x, \pm L, z)\boldsymbol{n} = \boldsymbol{v}(x, y, \pm L)\boldsymbol{n} = 0,$$

$$\int\limits_{\partial D_i} \boldsymbol{vn}\, d\boldsymbol{x} = 0, \ i = 1, 2, \ldots, N \Big\}.$$

Following [3], we obtain the lower-bound estimate

$$A \geqslant -\int\limits_Q \frac{1}{2} \boldsymbol{v}^2\, d\boldsymbol{x} + \int\limits_{z=\pm 1} \varphi^0 \boldsymbol{vn}\, d\boldsymbol{x} \qquad \forall \boldsymbol{v} \in W_p \quad (\operatorname{div} \boldsymbol{v} = 0). \tag{2.2}$$

Here and below, the function $\varphi^0(z)$ [such that $\varphi^0(\pm 1) = \pm 1$] is used for abbreviated notation and appears only in integrals over the faces $z = \pm 1$.

For the solution φ of problem (1.6), (1.7), the equality $I(\varphi) = J(\boldsymbol{v})$ holds, where $\boldsymbol{v} = \nabla\varphi$ and $J(\boldsymbol{v}) = -\int\limits_Q \frac{1}{2} \boldsymbol{v}^2\, d\boldsymbol{x} + \int\limits_{z=\pm 1} \varphi^0 \boldsymbol{vn}\, d\boldsymbol{x}$. This equality follows from the definition of the functionals $I(\varphi)$, $J(\boldsymbol{v})$, Green's formulas, and boundary conditions (1.3) and (1.5).

3. Formal Network Model. Solutions of the problem of the form (1.1), (1.2) for a pair of particles are known from electrostatics (see, for example, [7]). An analysis of these solutions shows that in many cases the approach of particles gives rise to strong fluxes between them; in this case, the energy is concentrated in the small domain (channel) between the particles. In view of this, we construct a network analogue of problem (1.1)–(1.5) assuming that the particles interact only with the nearest neighbors and the flux p_{ij} between the pair of adjacent particles (ith and jth) is equal to $C_{ij}^{(2)}(t_i - t_j)$, where $C_{ij}^{(2)}$ is the pair electric capacity of these particles in $\mathbb{R}^3$ [7]. We obtain a network (graph) $\{\boldsymbol{x}_i, t_i, C_{ij}^{(2)}; i, j = 1, 2, \ldots, N\}$, where $\boldsymbol{x}_i$ are the network nodes (particles), t_i are the potentials of the particles, and $C_{ij}^{(2)}$ are the characteristics of the network edges. The fluxes p_{ij} (charges, following the electrostatic interpretation of the problem) in the networks should satisfy Kirchhoff's equation for the internal nodes of the network (denoted below by I) and the boundary conditions for the particles lying on the boundaries $S^{\pm}$ corresponding to $z = \pm 1$:

$$\sum_{j=1}^{N} C_{ij}^{(2)}(t_i - t_j) = 0, \qquad i \in I, \quad t_i = \pm 1, \quad i \in S^{\pm}. \tag{3.1}$$

We define the notion of adjacent particles using the Voronoi–Delaunay method [8]. The Voronoi cell corresponding to a particle is a set of points that are closer to the given particle than to the remaining particles. For simply connected convex particles with piecewise-smooth boundaries, the Voronoi cells are determined uniquely. Adjacent particles are particle that lie in adjacent Voronoi cells (see Fig. 1). Accordingly, in (3.1), the summation should be performed only over the particles adjacent to the given particles (or one should assume that $C_{ij}^{(2)} = 0$ if the ith and jth particles are not adjacent). Some of the particles can be in contact with the boundaries $z = \pm 1$. On such boundaries, the Dirichlet condition is imposed. Some of the particles lie near the boundaries $z = \pm 1$, and their Voronoi cells have a part in common with the boundary $S^{\pm}$ and split $S^{\pm}$ into polyhedra (Fig. 1). We shall call these polyhedra pseudoparticles and specify potentials 1 or -1 on them. Via pseudoparticles we take into account the flux in the system boundary–near-boundary particle, which generates a third boundary condition. A pseudoparticle can be treated as a sphere of radius $R = \infty$. The Delaunay graph (a graph with the edges connecting adjacent particles) is connected [8].

All particles (initial and pseudoparticles) that intersect the boundary $z = 1$ will be denoted by S^+, all particles that intersect the boundary $z = -1$ by S^-, and the remaining (i.e., internal) particles by I. The following statement can be proved (see [3]).

Lemma 1. *The solution of problem* (3.1) *satisfies the condition* $-1 \leqslant t_i \leqslant 1$ $(i = 1, 2, \ldots, N)$.

Using the solution of problem (3.1), it is possible to construct bilateral estimates that, under certain conditions, are joined for close particle packing. We begin with obtaining refined [compared to (2.1) and (2.2)] estimates that are valid for any particle packing.

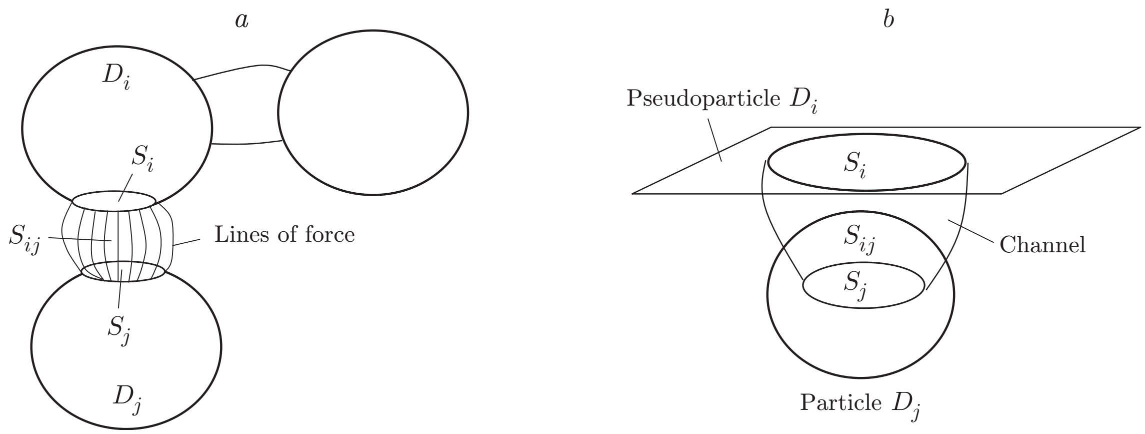

Fig. 2. Equipotential channel for a particle–particle pair (a) and a particle-pseudoparticle pair (boundary) (b).

4. Refined Lower-Bound Estimate. The estimates are refined using special trial functions. In (2.2), the trial function $\boldsymbol{v}$ should satisfy the conditions

$$\operatorname{div} \boldsymbol{v} = 0 \qquad \text{in the field of } Q; \tag{4.1}$$

$$\int\limits_{\partial D_i} \boldsymbol{vn}\, d\boldsymbol{x} = 0, \qquad i = 1, 2, \ldots, N; \tag{4.2}$$

$$\boldsymbol{vn} = 0 \qquad \text{on surfaces} \quad y = \pm L, \quad z = \pm L.$$

To construct the trial function, we consider two adjacent particles D_i and D_j (Fig. 2a). The choice of the direction of the coordinate axes is of no significance. The domain between the particles D_i and D_j (Fig. 2a and b) will be called the channel between the particles D_i and D_j and denoted by S_{ij}. In this study, we use two types of channels, which will be described below.

Equipotential Channels and Estimate of the Volume Integral in (2.2). A particle can have several neighbors. We choose the width of the channel S_{ij} (Fig. 2a) such that the channel is not intersected by other channels and, at the same time, its width S is different from zero.

We construct the channel S_{ij} and the trial function $\boldsymbol{v}$ in it based on the solution of the problem on the electric field in $\mathbb{R}^3$ produced by two particles D_i and D_j with potentials t_i and t_j, respectively:

$$\begin{gathered} \Delta\varphi = 0 \qquad \text{in the domain} \quad \mathbb{R}^3 \setminus (D_i \cup D_j), \\ \varphi = t_i \text{ on } D_i, \qquad \varphi = t_j \quad \text{on } D_j, \qquad |\varphi(\boldsymbol{x})| \to 0 \text{ at } |\boldsymbol{x}| \to \infty. \end{gathered} \tag{4.3}$$

The construction is performed in terms of the equipotential surfaces and lines of force. The surfaces $\varphi(\boldsymbol{x})$ = const are called equipotential surfaces and the normals to them form the lines of force. Outside $D_i \cup D_j$, the equipotential surfaces and the lines of force form a system of orthogonal coordinates.

For two particles, the structure of the equipotential surfaces and the lines of force is known and has the form shown in Fig. 2a. The lines of force issuing from the vicinity S_i of the pole of the sphere D_i are terminated in the vicinity S_j of the pole of the sphere D_j. By the poles is meant the nearest points of the particles. The domain S_{ij} filled with the lines of force passing from S_i to S_j and the domains S_i and S_j have dimensions of the same order of magnitude. We denote by δ_{ij} the distance between the particles D_i and D_j. For small δ_{ij} (tending to zero) and small (but fixed) dimensions of the domains S_i and S_j the domain S_{ij} forms an isolated channel between the particles D_i and D_j. Channels of such form will be called equipotential channels.

Let us consider the function

$$\boldsymbol{v} = \begin{cases} \nabla\varphi, & \boldsymbol{x} \in S_{ij}, \\ 0, & \boldsymbol{x} \in \mathbb{R}^3 \setminus S_{ij} \setminus (D_i \cup D_j). \end{cases} \tag{4.4}$$

Proposition. *For function* (4.4)*, the equality* $\operatorname{div} \boldsymbol{v} = 0$ *in* $\mathbb{R}^n(D_i \cup D_j)$ $(n = 2, 3)$ *holds.*

We note that for an arbitrary (not equipotential) channel in (4.4), this statement is incorrect. For the function (4.4),

$$\boldsymbol{v}\boldsymbol{n} = \nabla\varphi\boldsymbol{n} = 0 \qquad \text{on } \Gamma^+ \tag{4.5}$$

(Γ^+ is the lateral inner boundary of the channel S_{ij}). Equality (4.5) explains the choice of the channel S_{ij}. The boundary of the equipotential channel is formed of lines of force; therefore, there is no flux through the boundary and for function (4.4), the condition of zero divergence over the entire domain $\mathbb{R}^3 \setminus (D_i \cup D_j)$ is satisfied. If the electrostatic solution is used as a trial function, the problem of divergence of the integral does not arise.

Next, it is necessary to satisfy condition (4.2). By virtue of the choice of the function φ and the channel S_{ij} [see (4.5)], for the function $\boldsymbol{v}$ in (4.4) we have

$$\int\limits_{S_i} \nabla\varphi\boldsymbol{n}\,d\boldsymbol{x} = -\int\limits_{S_j} \nabla\varphi\boldsymbol{n}\,d\boldsymbol{x}. \tag{4.6}$$

Multiplying the equality $\Delta\varphi = 0$ from (4.3) by φ and integrating the result by parts in S_{ij} taking into account the remaining conditions, from (4.3) we obtain

$$-\int\limits_{S_{ij}} |\nabla\varphi|^2\,d\boldsymbol{x} = t_i \int\limits_{S_i} \nabla\varphi\boldsymbol{n}\,d\boldsymbol{x} + t_j \int\limits_{S_j} \nabla\varphi\boldsymbol{n}\,d\boldsymbol{x}.$$

From this, using (4.6) we obtain $\int\limits_{S_{ij}} |\nabla\varphi|^2\,d\boldsymbol{x} = (t_i - t_j) \int\limits_{S_i} \nabla\varphi\boldsymbol{n}\,d\boldsymbol{x}$. Then,

$$\int\limits_{S_i} \nabla\varphi\boldsymbol{n}\,d\boldsymbol{x} = \frac{1}{t_i - t_j} \int\limits_{S_{ij}} |\nabla\varphi|^2\,d\boldsymbol{x}. \tag{4.7}$$

Let us denote by $\varphi^{\pm 1}(\boldsymbol{x})$ the solution of problem (4.3) for $t_i = 1/2$ and $t_j = -1/2$. Then, $\nabla\varphi = (t_i - t_j)\varphi^{\pm 1}$ and relation (4.7) becomes

$$\int\limits_{S_i} \nabla\varphi\boldsymbol{n}\,d\boldsymbol{x} = (t_i - t_j) \int\limits_{S_{ij}} |\nabla\varphi^{\pm 1}|^2\,d\boldsymbol{x}. \tag{4.8}$$

Definition 2. The quantity $C^{S_{ij}} = \int\limits_{S_{ij}} |\nabla\varphi^{\pm 1}|^2\,d\boldsymbol{x} = \int\limits_{S_j} \nabla\varphi^{\pm 1}\boldsymbol{n}\,d\boldsymbol{x}$ will be called the capacity of the sets S_i and S_j (or D_i and D_j) in the set S_{ij}.

The capacity depends on the particle shape and the distance between the particles.

We calculated the flux $\int\limits_{S_i} \nabla\varphi\boldsymbol{n}\,d\boldsymbol{x}$ into the particle D_i through one channel S_{ij}. Since the particle D_i can have several neighbors, the integral in (4.2) is the total flux over all channels (which, by construction, do not intersect each other) that lead to D_i. To construct a trial function, it is necessary to specify the fluxes in each channel in such a manner that the flux balance holds immediately for all particles. For this, we use the solution of the network problem (3.1) — the quantities p_{ij} that satisfy the global balance condition. Condition (4.2) is satisfied if the flux in each channel S_{ij} satisfies the equality $\int\limits_{S_i} \nabla\varphi\boldsymbol{n}\,d\boldsymbol{x} = p_{ij}$. We assume that in the channel S_{ij}, the function φ has the form

$$\varphi = \lambda_{ij}\varphi^{\pm 1}. \tag{4.9}$$

From (4.9) it follows that in order that relation (4.8) be valid, λ_{ij} should satisfy the condition

$$\int\limits_{S_i} \lambda_{ij}\nabla\varphi^{\pm 1}\boldsymbol{n}\,d\boldsymbol{x} = \lambda_{ij} \int\limits_{S_i} \nabla\varphi^{\pm 1}\boldsymbol{n}\,d\boldsymbol{x} = p_{ij}$$

(S_i is the vicinity of the pole of the particle D_i) or

$$\lambda_{ij} C^{S_{ij}} = p_{ij}. \tag{4.10}$$

Bearing in mind that $p_{ij} = C_{ij}^{(2)}(t_i - t_j)$ ($C_{ij}^{(2)}$ is the capacity of the pair of bodies D_i and D_j in $\mathbb{R}^3$), from (4.8) and (4.10) we obtain

$$\lambda_{ij} = \frac{C_{ij}^{(2)}}{C^{S_{ij}}}\,(t_i - t_j). \tag{4.11}$$

If the trial function in the channel S_{ij} has the form (4.9), condition (4.2) is satisfied for it. The satisfaction of (4.1) was proved earlier (it does not depend on the multiplication of the function by an arbitrary number).

We can now calculate the integral in (2.2) for the trial function (4.9). Let us calculate the value of the integral in one channel. For function (4.9) which satisfies (4.1) and (4.2), we have

$$\int\limits_{S_{ij}} |\lambda_{ij}\nabla\varphi^{\pm 1}|^2\,d\boldsymbol{x} = \lambda_{ij}^2 \int\limits_{S_{ij}} |\nabla\varphi^{\pm 1}|^2\,d\boldsymbol{x} = \lambda_{ij}^2 C^{S_{ij}}. \tag{4.12}$$

In view of (4.11), the value of (4.12) (integral over one channel S_{ij}) is equal to

$$\lambda_{ij}^2 C^{S_{ij}} = \Big(\frac{C_{ij}^{(2)}}{C^{S_{ij}}}\Big)^2 C^{S_{ij}}(t_i - t_j)^2 = \frac{(C_{ij}^{(2)})^2}{C^{S_{ij}}}\,(t_i - t_j)^2. \tag{4.13}$$

Since the channels do not intersect each other, the integral $-\int\limits_Q \frac{1}{2}\boldsymbol{v}^2\,d\boldsymbol{x}$ is equal to the sum over all channels. Taking into account (4.12) and (4.13), we obtain

$$-\int\limits_Q \frac{1}{2}\boldsymbol{v}^2\,d\boldsymbol{x} = -\frac{1}{2}\sum_{S_{ij}} \frac{(C_{ij}^{(2)})^2}{C^{S_{ij}}}\,(t_i - t_j)^2 = -\frac{1}{4}\sum_{i,j=1}^{N} \frac{(C_{ij}^{(2)})^2}{C^{S_{ij}}}\,(t_i - t_j)^2.$$

The index S_{ij} in the summation sign indicates that the summation is performed over the channels (one channel refers to one term). Performing the summation over the subscripts $i, j = 1, 2, \ldots, N$, we pass twice through each channel and, as a result, obtain a multiplier of 1/4.

Estimate of the Boundary Integral in (2.2). For a particle adjacent to the boundary $z = 1$, the channel S_{ij} has the shape shown in Fig. 2b. The pseudoparticle D_i is plane. The field in the channel is calculated as in the previous case (one only needs to set $R_i = 0$). The boundary integral from (2.2) over S_i is equal to p_{ij}, and the integral over the entire boundary $z = 1$ is equal to $\sum\limits_{i\in S^+}\sum\limits_{j} p_{ij} = \sum\limits_{i\in S^+}\sum\limits_{j} C_{ij}^{(2)}(1 - t_j)$, where the subscript $i \in S^+$ corresponds to the particles (true particles and pseudoparticles) lying on the boundary $z = 1$. We denote this sum by P^+. Similarly, $P^- = \sum\limits_{i\in S^-}\sum\limits_{j} C_{ij}^{(2)}(-1 - t_j)$. Thus, both integrals in (2.2) are calculated for the trial function (4.3) we can write the estimate implied by (2.2):

$$A \geqslant \frac{1}{4}\sum_{i,j=1}^{N} \frac{(C_{ij}^{(2)})^2}{C^{S_{ij}}}\,(t_i - t_j)^2 + P^+ + P^-. \tag{4.14}$$

5. Refined Upper-Bound Estimate. The general upper-bound estimate has the form (2.1). Let us refine it.

Cylindrical Channel and Constructing a Trial Function. We consider a cylindrical channel R_{ij} (Fig. 3) and define in it a trial function $\varphi = (t_i - t_j)\varphi^{\pm 1}$ (restriction of the electrostatic solutions to the channel R_{ij}). Of interest is the integral $\int\limits_Q |\nabla\varphi|^2\,d\boldsymbol{x}$. For $\varphi = (t_i - t_j)\varphi^{\pm 1}$, in the channel R_{ij} we have

$$\int\limits_{R_{ij}} |\nabla\varphi|^2\,d\boldsymbol{x} = (t_i - t_j)^2 \int\limits_{R_{ij}} |\nabla\varphi^{\pm 1}|^2\,d\boldsymbol{x}.$$

Constructing a Trial Function outside the Channel. We consider a particle with channels exiting R_{ij}. The diameter of the channels R_{ij} is fixed. With approach of the channels ($\delta \to 0$), the distances between the particles D_i and the distances between the channels R_{ij} are separated from zero uniformly over $\delta \to 0$. We construct a vicinity of the particles D_i and channels R_{ij} of width h which does not depend on $\delta \to 0$. We denote this vicinity by P_h. As $\delta \to 0$, the domain P_h passes to its limiting (at $\delta = 0$) position. In the domain P_h, we construct a function $\psi^\delta(\boldsymbol{x})$

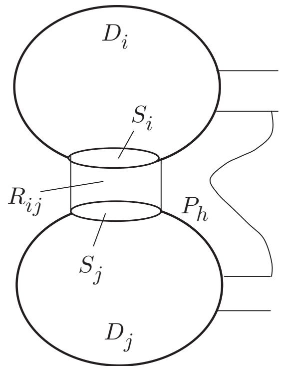

Fig. 3. Cylindrical channel and the vicinity of P_h.

that takes specified values on the boundaries of the particles D_i and D_j and the boundary of the channel R_{ij} and vanishes on the boundary γ. The question consists of the possibility of constructing such a function subject to the additional condition that its derivatives are bounded uniformly over δ. This question is solved positively by virtue of the results of [9, Chapter 15].

The function

$$\varphi(\boldsymbol{x}) = \begin{cases} \varphi^{\pm 1} & \text{in } R_{ij}, \\ \psi^{\delta} & \text{in } P_h, \\ 0 & \text{outside of } P_h \end{cases} \tag{5.1}$$

belongs to the set V_p and can be used as a trial function in (2.1). For (5.1), we write

$$\int\limits_Q |\nabla\varphi|^2\,d\boldsymbol{x} = \frac{1}{2}\sum_{i,j=1}^{N}(t_i - t_j)^2 C^{R_{ij}} + \int\limits_{P_h} |\nabla\psi^{\delta}(\boldsymbol{x})|^2\,d\boldsymbol{x}, \tag{5.2}$$

where $C^{R_{ij}} = \int\limits_{R_{ij}} |\nabla\varphi^{\pm 1}|^2\,d\boldsymbol{x}$. In this case, the integral in (5.2) satisfies the inequality

$$\int\limits_{P_h} |\nabla\psi^{\delta}(\boldsymbol{x})|^2\,d\boldsymbol{x} \leqslant C < \infty, \tag{5.3}$$

where C does not depend on δ.

6. Asymptotic Shielding. Approximation of the Continuous Problem by a Discrete Problem for Shielding Particles. According to [6], the essence of the asymptotic shielding effect lies in the absence of the effect of other bodies on the mutual capacity of two closely spaced bodies. In the presence of the shielding effect for all pairs of adjacent particles, the obtained refined bilateral estimates for the effective conductivity A are joined in the asymptotic (for $\delta \to 0$) sense and the main element A is expressed in terms of the solution of the network problem (3.1).

The notion of neighbors was defined above. We introduce the packing parameter $\delta = \max \delta_{ij}$ (δ_{ij} is the distance between the ith and jth particles; the maximum is taken only over adjacent particles). Pseudoparticles also are included in the consideration. The condition $\delta \to 0$ implies that the particles approach each other and the near-boundary particles approach the boundary. This situation can be called close particle packing.

According to estimates (4.14), (5.2), and (5.3), we have

$$A \geqslant -\frac{1}{4}\sum_{i,j=1}^{N}\frac{(C_{ij}^{(2)})^2}{C^{S_{ij}}}(t_i - t_j)^2 + P^+ + P^-, \quad A \leqslant \frac{1}{4}\sum_{i,j=1}^{N} C^{R_{ij}}(t_i - t_j)^2 + \int\limits_{P_h} |\nabla\psi^{\delta}|^2\,d\boldsymbol{x}. \tag{6.1}$$

Lemma 2. *The equality*

$$\frac{1}{2}\sum_{i,j=1}^{N} C_{ij}^{(2)}(t_i - t_j)^2 = P^+ + P^- \tag{6.2}$$

holds, where $\{t_i\}$ *is the solution of problem* (3.1).

The proof of Lemma 2 is similar to that given in [3].

By virtue of (6.1) and (6.2), we have the estimate

$$\frac{1}{2}\sum_{i,j=1}^{N} C_{ij}^{(2)}\Big[-\frac{1}{2}\frac{(C_{ij}^{(2)})^2}{C^{S_{ij}}}+1\Big](t_i-t_j)^2 \leqslant A \leqslant \frac{1}{4}\sum_{i,j=1}^{N} C^{S_{ij}}(t_i-t_j)^2+\int\limits_{P_h}|\nabla\psi^{\delta}|^2\,d\boldsymbol{x}. \tag{6.3}$$

Let there be two particles D_i and D_j in $\mathbb{R}^3$ separated by a distance δ. We distinguish the channel K that connects the particle. By the channel K is meant the channel S_{ij} or R_{ij}.

Lemma 3 (on shielding)**.** *As $\delta \to 0$, for the particles described above:*

1) *the energy outside the channel is* $\displaystyle\int\limits_{\mathbb{R}^3\setminus(K\cup D_i\cup D_j)}|\nabla\varphi^{\pm 1}|^2\,d\boldsymbol{x}\leqslant C<\infty$*, where C does not depend on δ;*

2) *if for any adjacent particles D_i and D_j, the condition* $\displaystyle\int\limits_{\mathbb{R}^3\setminus(D_i\cup D_j)}|\nabla\varphi^{\pm 1}|^2\,d\boldsymbol{x}\to\infty$ *is satisfied for $\delta\to 0$, the capacities $C_{ij}^{(2)}$, $C^{S_{ij}}$, and $C^{R_{ij}}$ are asymptotically equivalent: $C_{ij}^{(2)}\sim C^{S_{ij}}\sim C^{R_{ij}}$.*

Proof. We write $f\sim g$ for $\delta \to 0$ if $f/g\to 1$ as $\delta\to 0$. The position of the particles D_i and D_j is not fixed, which creates some technical difficulties in obtaining estimates. However, the fact that the particles move to the known limiting position (contact position) reduces these difficulties considerably.

Estimate of the Energy outside the Sphere $|\boldsymbol{x}|=R$. We enclose two approaching particles D_i and D_j in a sphere of radius R and show that outside the sphere the energy is bounded uniformly over δ. On the sphere $|\boldsymbol{x}|=R$, where $|\varphi^{\pm 1}|\leqslant 1$ by virtue of the maximum principle irrespective of the position of the particles D_i and D_j. We write Poisson's integral [10] in the form

$$\varphi^{\pm 1}=\frac{1}{4\pi R}\int\limits_{|\boldsymbol{x}|=R}\frac{R^2-|\boldsymbol{x}|^2}{|\boldsymbol{x}-\boldsymbol{y}|^2}\,u(\boldsymbol{y})\,d\boldsymbol{y}, \tag{6.4}$$

where $u(\boldsymbol{y})=\varphi^{\pm 1}(\boldsymbol{y})$ is the value of the function $\varphi^{\pm 1}(\boldsymbol{y})$ on the sphere $|\boldsymbol{x}|=R$. From (6.4) it follows that $|\varphi^{\pm 1}|\leqslant C/|\boldsymbol{x}|$ [10], where C does not depend on the positions of D_i and D_j during their approach (D_i and D_j are in a sphere $|\boldsymbol{x}|\leqslant R$).

Differentiating (6.4) (for $|\boldsymbol{x}|\geqslant R$, it is possible to differentiate under the integral sign), we find that as $\rho\to\infty$,

$$\int\limits_{R\leqslant|\boldsymbol{x}|}|\nabla\varphi^{\pm 1}|^2\,d\boldsymbol{x}\leqslant C_1<\infty,$$

where C_1 does not depend on δ.

Estimate of the Energy inside the Sphere $|\boldsymbol{x}|=R$. Let us consider the set $M=\{|\boldsymbol{x}|\leqslant R\}\setminus(D_i\cup D_j\cup K)$ — a sphere $\{|\boldsymbol{x}|\leqslant R\}$ without the particles D_i and D_j and the channel K. For the domain M, we can use the results of [9, Chapter 15], by virtue of which the modulus $|\nabla\varphi^{\pm 1}(\boldsymbol{x})|$ is bounded by a quantity that does not depend on the positions of D_i and D_j during their approach. Hence, the integral of $|\nabla\varphi^{\pm 1}(\boldsymbol{x})|^2$ over $M=\{|\boldsymbol{x}|\leqslant R\}\setminus(D_i\cup D_j\cup R_{ij})$ is bounded uniformly over δ. Part 1 of Lemma 3 (on shielding) is proved. Let us now prove Part 2.

The asymptotic equivalence $C_{ij}^{(2)}\sim C^{R_{ij}}$ follows from the equality

$$C_{ij}^{(2)}=C^{R_{ij}}+\int\limits_{\mathbb{R}^3\setminus(R_{ij}\cup D_i\cup D_j)}|\nabla\varphi^{\pm 1}|^2\,d\boldsymbol{x}\leqslant C, \tag{6.5}$$

which follows from the definition of the quantities $C_{ij}^{(2)}$ and $C^{R_{ij}}$ [as integrals over $\mathbb{R}^3\setminus(D_i\cup D_j)$ and over R_{ij}] and the finiteness of the integral over $\mathbb{R}^3\setminus(R_{ij}\cup D_i\cup D_j)$ (according to Part 1 of Lemma 3).

If $\displaystyle\int\limits_{\mathbb{R}^3\setminus(D_i\cup D_j)}|\nabla\varphi^{\pm 1}|^2\,d\boldsymbol{x}\to\infty$ as $\delta\to 0$, then $C^{R_{ij}}\to\infty$ as $\delta\to 0$ because $\displaystyle\int\limits_{\mathbb{R}^3\setminus R_{ij}}|\nabla\varphi^{\pm 1}|^2\,d\boldsymbol{x}$ is uniformly bounded over δ (see Part 1 of Lemma 3). Dividing (6.5) by $C^{R_{ij}}$, we obtain

$$\frac{C_{ij}^{(2)}}{C^{R_{ij}}} = 1 + \frac{1}{C^{R_{ij}}} \int\limits_{\mathbb{R}^3 \setminus R_{ij}} |\nabla \varphi^{\pm 1}|^2 \, d\boldsymbol{x} \to 1 \qquad \text{at} \quad \delta \to 0.$$

Let us prove, that $C_{ij}^{(2)} \sim C^{S_{ij}}$. The channel S_{ij} is not cylindrical. Its width is separated from zero for at all positions of D_i and D_j as $\delta \to 0$. We circumscribe a cylinder $\mathcal{S}$ into S_{ij} and calculate the capacity $C^{\mathcal{S}} = \int\limits_{\mathcal{S}} |\nabla \varphi^{\pm 1}|^2 \, d\boldsymbol{x}$ with respect to this cylinder $\mathcal{S}$. Since the capacities $C^{\mathcal{S}}$, $C^{S_{ij}}$, and $C_{ij}^{(2)}$ are obtained by integration of the function $|\nabla \varphi^{\pm 1}|^2 > 0$ over the domains $\mathcal{S} \subset S_{ij} \subset \mathbb{R}^3 \setminus (D_i \cup D_j)$, the following relations hold:

$$C^{\mathcal{S}} \leqslant C^{S_{ij}} \leqslant C_{ij}^{(2)}, \tag{6.6}$$

For the cylindrical channel $\mathcal{S}$, the statement of Part 2 of Lemma 3 was proved above. By virtue of this, $C^{\mathcal{S}} \sim C_{ij}^{(2)}$, whence in view of (6.6), we infer that $C^{S_{ij}} \sim C_{ij}^{(2)}$.

To continue the analysis of the problem, it is necessary to address the physics of the phenomenon considered. As $\delta \to 0$, the pair capacities of the particles can have different orders in δ (this depends on the particle shape). We require that the particles be of the same sort, i.e., we require that as $\delta \to 0$, the pair capacities of the particles have the same order $f(\delta)$: $mf(\delta) \leqslant C_{ij}^{(2)} \leqslant Mf(\delta)$, where m and M do not depend on δ.

Let us define the energy in the discrete network as $E = \dfrac{1}{4} \sum\limits_{i,j=1}^{N} C_{ij}^{(2)} (t_i - t_j)^2$, where $\{t_i\}$ is a solution of the discrete network problem (3.1).

Lemma 4. *If as $\delta \to 0$, the capacities $C_{ij}^{(2)}$ have the same order $f(\delta)$, then E has the same order $f(\delta)$.*

The proof is similar to that given in [3].

Theorem 1. *Let the capacities $C_{ij}^{(2)}$ have the same order $f(\delta) \to \infty$ as $\delta \to 0$ for all adjacent particles. Then, the effective conductivity $A \to \infty$ as $\delta \to 0$. For $\delta \to 0$, the main element A is expressed in terms of $\{t_i\}$ — a solution of the discrete network problem* (3.1):

$$A \sim \frac{1}{4} \sum_{i,j=1}^{N} C_{ij}^{(2)} (t_i - t_j)^2. \tag{6.7}$$

Proof. By virtue of (6.3), we have

$$\frac{1}{2} \sum_{i,j=1}^{N} C_{ij}^{(2)} \Big[-\frac{1}{2} \frac{(C_{ij}^{(2)})^2}{C^{S_{ij}}} + 1 \Big] (t_i - t_j)^2 \leqslant A \leqslant \frac{1}{4} \sum_{i,j=1}^{N} C^{R_{ij}} (t_i - t_j)^2 + \int\limits_{P_h} |\nabla \psi^{\delta}|^2 \, d\boldsymbol{x}, \tag{6.8}$$

where $\{t_i\}$ is a solution of the discrete network problem that does not depend on δ; $\int\limits_{P_h} |\nabla \psi^{\delta}|^2 \, d\boldsymbol{x} < C < \infty$, where C does not depend on δ [see (5.3)]. We let δ to zero. By the condition of Theorem 1, the condition of Part 2 of Lemma 3 is satisfied, hence,

$$C_{ij}^{(2)} / C^{S_{ij}} \to 1, \qquad C^{R_{ij}} / C^{R_{ij}} \to 1. \tag{6.9}$$

By virtue of the first relation in (6.9), we have $-(1/2) C_{ij}^{(2)} / C^{S_{ij}} + 1 = (1/2) C_{ij}^{(2)} + O(\delta)$ [$O(\delta) \to 0$ as $\delta \to 0$]. From the second relation in (6.9) it follows that $C^{R_{ij}} = C_{ij}^{(2)} + C_{ij}^{(2)} O(\delta)$. Then, (6.9) is written as

$$\frac{1}{4} \sum_{i,j=1}^{N} C_{ij}^{(2)} (t_i - t_j)^2 + 2 \max \| C_{ij}^{(2)} \| O(\delta) \leqslant A$$

$$\leqslant \frac{1}{4} \sum_{i,j=1}^{N} C_{ij}^{(2)} (t_i - t_j)^2 + 2 \max \| C_{ij}^{(2)} \| O(\delta) + \int\limits_{P_h} |\nabla \psi^{\delta}|^2 \, d\boldsymbol{x}. \tag{6.10}$$

Here we used the inequality $|t_i| \leqslant 1$ (see Lemma 1).

Dividing both parts (6.10) by $E = \frac{1}{4}\sum_{i,j=1}^{N} C_{ij}^{(2)}(t_i - t_j)^2$, we obtain

$$1 + \frac{2\|C_{ij}^{(2)}\|}{E} O(\delta) \leqslant \frac{A}{E} \leqslant 1 + \frac{2\|C_{ij}^{(2)}\|}{E} O(\delta) + \frac{1}{E}\int\limits_{P_h} |\nabla\psi^\delta| \, d\boldsymbol{x}. \tag{6.11}$$

By the condition of the theorem, $C_{ij}^{(2)}$ have the same order $f(\delta) \to \infty$ as $\delta \to 0$. By virtue of Lemma 4, $E \to \infty$ as $\delta \to 0$ and the integral $\int\limits_{P_h} |\nabla\psi^\delta|^2 \, d\boldsymbol{x}$ is bounded uniformly over δ. Then, the right and left parts of (6.11) tend to unity as $\delta \to 0$, whence follows (6.7).

7. Discussion of the Results. The region of application of the above results is fairly wide because of the extensive use of network models [11] and high-contrast composites [2, 3]. We give the basic conclusions from the mathematical results obtained above.

Asymptotic Shielding Effect. Tamm explains the asymptotic shielding effect (without using the term asymptotic) in such a manner: "... if the conductor dimensions are larger than the distance between the conductors ... the space between the capacitor plates is considerably (if not completely) protected by the plates from the effect of the external field" [6, p. 53]. In the three-dimensional case, the shielding effect may not occur. The condition of its occurrence is described in the shielding lemma. The physics of the phenomenon is the following. In the case of unbounded increase in the capacity with approach of the particles, the energy channeling effect is observed (by virtue of Part 1 of Lemma 3, the increasing energy is concentrated in the narrow channel between the particles). The flux in the channel is protected from the effect of the external field by its large magnitude and not by the bodies (the situation more resembles the stress concentration effect [12] than the classical shielding case, where the bodies are protected by the shield).

Effective Conductivity of High-Contrast Composites and Asymptotic Shielding Effect. In developing technologies involving high-contrast composites, one usually reasons as follows: the particles are highly conducting and the distance between them is small, hence, one might expect a large total flux through the composite. This reasoning is generally incorrect since the effective conductivity is determined by pair capacities rather than by contrast. To raise the effective conductivity, it is necessary to increase the pair capacities of the particles which are determined primarily by the particle geometry.

Effect of the Dimension of the Problem. The above results are true for problems of any dimension. The dimension influence only the pair capacities, whose difference can lead to a difference in the properties of the composites depending on the dimension of the problem.

Network Models. A finite-dimensional (network) approximation for continuous high-contrast problems is possible in the presence of the asymptotic shielding effect for pairs of particles. In the absence of this effect, the use of network models is generally incorrect. The presence of the shielding effect is verified by calculations of the pair capacities of the particles.

Order of the Pair Conductivities and Particle Shape. In the problem in question there is the order of the pair capacities $f(\delta)$ present. In particular, the pair capacities define the effective conductivity of a composite. We gives values of the pair capacities in $\mathbb{R}^n$ $(n = 2, 3)$ for some characteristic cases.

We first consider three-dimensional pairs.

1. Sphere–sphere. The pair capacity of two spheres of radius R separated by a distance δ is equal to $4\pi \ln(\delta/R) \to \infty$ as $\delta \to 0$ [5, 10].

2. Plane–cone. For the formation of a plane–cone pair (polyhedral angle), $f(\delta)$ is bounded as $\delta \to 0$. The shielding condition is not satisfied.

3. For a polyhedron–polyhedron pair in general position, $f(\delta)$ is bounded as $\delta \to 0$. A face–face pair (plate capacitor) can form, for which $f(\delta) \sim 1/\delta \to \infty$ as $\delta \to 0$. For random particle packing, the occurrence of face–face pairs is improbable.

4. A pair of particles whose shape in the region of contact (a is the characteristic dimension of the domain) is given by $z = \pm r^4/2$, where $r = \sqrt{x^2 + y^2}$. This function is more suitable for modeling relatively flat particle

fragments. The capacity $C^{(2)} = \int\limits_0^a \frac{r\,dr}{\delta + r^4} = \frac{1}{\sqrt{\delta}} \arctan \frac{a^2}{\delta} \sim \frac{\pi}{\delta} \to \infty$ as $\delta \to 0$; for $\delta \to 0$, $C^{(2)}$ does not depend on a.

Let us now consider two-dimensional pairs.

5. Disk–disk. The pair capacity of two disks R separated by a distance δ is equal to $\pi\sqrt{R/\delta} \to \infty$ as $\delta \to 0$ [7]. From this directly follow the results of [3].

6. Half-plane–corner. The pair capacity of a half-plane and an angle separated by a distance δ has order $\ln\delta \to \infty$ as $\delta \to 0$. We note that the integral $\int\limits_0^\infty \frac{dx}{\delta + |x|}$ diverges and this the case cannot be investigated by the methods of [3].

Let us compare the pair capacity of a disk–disk pair (denoted by the subscript d–d) and a half-plane–corner pair (the subscript h–c). In the two-dimensional case, $C^{(2)}_{\text{d–d}}/C^{(2)}_{\text{h–c}} \cong \delta^{-1/2}/\ln\delta \to \infty$ as $\delta \to 0$, i.e., the disks are more effective fillers for increasing the conducting properties of the composites than polyhedra (which form a half-plane–corner pair in the region of approach). In the three-dimensional case, the difference is even larger (cf. the sphere–sphere and plane–cone cases). We note that the particles having the shape of polyhedra are widely used because of the low cost of their production by crushing (usually by grinding).

Notion of Capacity. In the present paper, the problem was considered from the viewpoint of electrostatics [6, 7]. Problem (1.1)–(1.5) describes the processes of electrical and thermal conduction and diffusion, for which the notion of capacity, as a rule, is not used.

REFERENCES

1. J-L. Lions, "Some computational aspects of the homogenization method in composite materials," in: *Computational Methods in Mathematics, Geophysics, and Optimum Control* [Russian translation], Nauka, Novosibirsk (1978), pp. 5–19.
2. L. Borcea and G. Papanicolaou, "Network approximation for transport properties of high contrast materials," *SIAM J. Appl. Math.*, **58**, No. 2, 501–539 (1998).
3. L. V. Berlyand and A. G. Kolpakov, "Network approximation in the limit of small interparticle distance of the effective properties of a high contrast random dispersed composite," *Arch. Rational Mech. Anal.*, No. 3, 179–227 (2001).
4. J. C. Maxwell, *Treatise on Electricity and Magnetismus*, Vol. 1, Oxford Univ. Press, Oxford (1873).
5. J. B. Keller, "Conductivity of a medium containing a dense array of perfectly conducting spheres or cylinders or nonconducting cylinders," *J. Appl. Phys.*, **34**, No. 4, 991–993 (1963).
6. I. E. Tamm, *Foundations of Electricity Theory* [in Russian], Nauka, Moscow (1976).
7. W. R. Smythe, *Static and Dynamic Electricity*, McGraw-Hill, New York–Toronto–London (1950).
8. N. N. Medvedev, *Vorornoi–Delaunay Method and the Structure of Noncrystalline Systems* [in Russian], Izd. Sib. Otd. Ross. Akad. Nauk, Novosibirsk (2000).
9. O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptical Equations* [in Russian], Nauka, Moscow (1964).
10. V. S. Vladimirov, *Equations of Mathematical Physics* [in Russian], Nauka, Moscow (1976).
11. M. Sahimi, *Heterogeneous Materials*, Springer-Verlag, Berlin–Heidelberg–New York (2003).
12. Yu. N. Rabotnov, *Mechanics of Deformable Solids* [in Russian], Nauka, Moscow (1979).